

# **Maximum Simulated Likelihood Estimation of the Seemingly Unrelated Stochastic Frontier Regressions**

Hung-pin Lai  
National Chung Cheng University

November 26, 2020

# Outline

- **Introduction: Motivation and main contribution**
- **The model:**
  - I. The cross-sectional model
  - II. The panel model with random effects
- **Model estimation:**
  - I. The MLE
  - II. The simulated MLE
- **Simulation**
- **Empirical application**

# Introduction

- If an organization (a DMU) operates multiple production divisions (sub-DMUs), with each division supported by its own set of resource inputs, **these sub-DMUs may be subject to the same random shocks** as the parent DMU.
- Given that these divisions would share some commonly observed or unobserved characteristics of the parent DMU, the divisions' technical efficiencies may well be correlated.
- Since the system estimation takes into account the **mutual dependency among the composite errors**, the estimator is more efficient than that from the regression-by-regression estimation.

- The same firm may well share some commonly observed or unobserved characteristics, while there may also be correlations between their technical inefficiency levels
- The multi-sector SF model under consideration is similar to the multiple-output model.
- The cost frontier approach -- under many circumstances only the input and output variables are observed empirically.

## Objective and contribution of this paper

- ✓ Propose a system of panel SF model for the multi-sector analysis
- ✓ Use a copula-based maximum simulated likelihood estimation method
- ✓ Monte Carlo simulation
- ✓ Empirical study: Taiwan hotel data, including the accommodation and restaurant sectors.

## The seemingly unrelated SF panel model

- ✓ Consider the following production frontiers of  $J$  sub-DMUs:

$$y_{it}^1 = \beta_0^1 + x_{it}^{1'} \beta_1 + \alpha_i^1 + v_{it}^1 - u_{it}^1,$$

$$y_{it}^2 = \beta_0^2 + x_{it}^{2'} \beta_2 + \alpha_i^2 + v_{it}^2 - u_{it}^2,$$

⋮

$$y_{it}^J = \beta_0^J + x_{it}^{J'} \beta_J + \alpha_i^J + v_{it}^J - u_{it}^J,$$

where  $j = 1, \dots, J$ ,  $i = 1, \dots, N$ , and  $t = 1, \dots, T$ .  $y_{it}^j$  and  $x_{it}^j$  are the log output and the log inputs of the  $j$ th sub-DMU.

## Main assumptions about the random components:

[A1]  $\alpha_i^j \sim N(0, \sigma_{\alpha j}^2)$  is the firm-specific random effect.

For a fixed  $i$ ,  $\alpha_i^j$  and  $\alpha_i^{j'}$  are independent to each other for  $j \neq j'$ .

[A2]  $v_{it}^j \sim N(0, \sigma_{v j}^2)$  is a two-sided symmetric random noise.

For fixed  $i$  and  $j$ ,  $v_{it}^j$  and  $v_{is}^{j'}$  are independent across time.

For fixed  $i$  and  $t$ ,  $v_{it}^j$  and  $v_{it}^{j'}$  are correlated for  $j \neq j'$ .

**[A3]**  $u_{it}^j \sim N^+(0, \sigma_{uj,it}^2)$  is a one-sided random component that captures the inefficiency and its variance.

$\sigma_{uj,it}$  can be parametrized as  $\sigma_{uj,it} = \exp(\delta'_j w_{it}^j)$

For fixed  $i$  and  $j$ ,  $u_{it}^j$  and  $u_{is}^{j'}$  are independent across time.

For fixed  $i$  and  $t$ ,  $u_{it}^j$  and  $u_{it}^{j'}$  are correlated for  $j \neq j'$ .

**[A4]** The three random components  $\alpha_i^j$ ,  $v_{it}^j$  and  $u_{it}^j$  are independent to each other and uncorrelated with  $x_{it}$  for fixed  $i$ ,  $t$ , and  $j$ .

# Copulas and the simulated likelihood function

## I. The model without random effects

- ✓ Let  $F_{\varepsilon^j}(\varepsilon_{it}^j; \theta_j)$  and  $f_{\varepsilon^j}(\varepsilon_{it}^j)$  denote the cumulative distribution function (cdf) and probability density function (pdf) of  $\varepsilon_{it}^j$ , respectively.
- ✓ If  $u \sim N^+(0, \sigma_{uj}^2)$  and  $v \sim N(0, \sigma_{vj}^2)$ , then the marginal pdf of  $\varepsilon_{it}^j$  is

$$f_{\varepsilon^j}(\varepsilon_{it}^j) = \frac{2}{\sigma_j} \phi\left(\frac{\varepsilon_{it}^j}{\sigma_j}\right) \Phi\left(-\frac{\lambda_j}{\sigma_j} \varepsilon_{it}^j\right)$$

where  $\phi(\cdot)$  represents a standard normal density function,

$$\sigma_j = \sqrt{\sigma_{vj}^2 + \sigma_{uj,it}^2} \text{ and } \lambda_j = \sigma_{uj,it}^2 / \sigma_{vj}^2.$$

- ✓ In other words,  $\varepsilon_{it}^j$  follows a **closed skew normal distribution**, i.e.,

$$\varepsilon_{it}^j \sim CSN_{1,1} \left( 0, \sigma_j^2, -\frac{\lambda_j^2}{1 + \lambda_j^2}, 0, \frac{\lambda_j^2 \sigma_j^2}{(1 + \lambda_j^2)^2} \right)$$

which has the cdf

$$F_{\varepsilon^j}(\varepsilon_{it}^j) = 2 \cdot \Phi_2 \left( \begin{pmatrix} \varepsilon_{it}^j \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_j^2 & \frac{\lambda_j^2 \sigma_j^2}{1 + \lambda_j^2} \\ \frac{\lambda_j^2 \sigma_j^2}{1 + \lambda_j^2} & \frac{\lambda_j^2 \sigma_j^2}{1 + \lambda_j^2} \end{pmatrix} \right)$$

where  $\Phi_2(\cdot; \mu, \Sigma)$  denotes the cdf of the bivariate normal distribution with mean  $\mu$  and variance  $\Sigma$ .

## Sklar's theorem in $J$ -dimension (Schweizer and Sklar, 1983)

- ✓ Let  $F(\cdot)$  be a  $J$ -dimensional distribution function with margins  $F_1(\cdot), \dots, F_J(\cdot)$ .
- ✓ Let  $R$  be the vector of parameters in the copula function, then  $\theta = (\theta_1^T, \dots, \theta_J^T, \theta_c^T)^T$  denotes the parameters contained in the joint CDF  $F(\cdot)$ .
- ✓ Then there exists a  $J$ -copula  $C(\cdot)$  such that for all  $\varepsilon_i = (\varepsilon_{1i}, \dots, \varepsilon_{Ji})^T$  in  $\bar{R}^J$

$$F_\varepsilon(\varepsilon_{it}^1, \dots, \varepsilon_{it}^J) = C(F_{\varepsilon^1}(\varepsilon_{it}^1), \dots, F_{\varepsilon^J}(\varepsilon_{it}^J); R).$$

- If  $F_1(\cdot), \dots, F_J(\cdot)$  are all continuous, then  $C(\cdot)$  is unique.

## The joint PDF

By the Sklar theorem, the joint pdf of  $\varepsilon_{it}$

$$f_\varepsilon(\varepsilon_{it}^1, \dots, \varepsilon_{it}^J) = c(F_{\varepsilon^1}(\varepsilon_{it}^1), \dots, F_{\varepsilon^J}(\varepsilon_{it}^J); R) \cdot \prod_{j=1}^J f_{\varepsilon^j}(\varepsilon_{it}^j).$$

where the copula density is defined as

$$c(F_{\varepsilon^1}(\varepsilon_{it}^1), \dots, F_{\varepsilon^J}(\varepsilon_{it}^J); R) = \frac{\partial^J C(F_{\varepsilon^1}(\varepsilon_{it}^1), \dots, F_{\varepsilon^J}(\varepsilon_{it}^J); R)}{\partial F_{\varepsilon^1}(\varepsilon_{it}^1) \dots \partial F_{\varepsilon^J}(\varepsilon_{it}^J)}.$$

## Examples of copula:

- ✓ The **independent** (or **product**) copula:

$$C(\zeta_{it}^1, \dots, \zeta_{it}^J) = \prod_{j=1}^J \zeta_{it}^j,$$

where  $\zeta_{it}^j = F_{\varepsilon^j}(\varepsilon_{it}^j)$ . The corresponding copula density is

$$c(\zeta_{it}^1, \dots, \zeta_{it}^J) = 1.$$

- ✓ The **Gaussian** copula:

$$C(F_{\varepsilon^1}(\varepsilon_{it}^1), \dots, F_{\varepsilon^J}(\varepsilon_{it}^J); R) = \Phi_R \left( \Phi^{-1} \left( F_{\varepsilon^1}(\varepsilon_{it}^1) \right), \dots, \Phi^{-1} \left( F_{\varepsilon^J}(\varepsilon_{it}^J) \right) \right),$$

where  $\Phi_R(\cdot)$  denotes the standardized multivariate normal distribution with correlation coefficient matrix  $R$ .

The corresponding copula density is

$$c(\zeta_{it}^1, \dots, \zeta_{it}^J) = \frac{1}{|R|^{1/2}} \exp \left( -\frac{1}{2} \zeta_{i \cdot}' (R^{-1} - I) \zeta_{i \cdot} \right),$$

where  $\zeta_{i \cdot} = (\zeta_{it}^1, \dots, \zeta_{it}^J)'$  and  $\zeta_{it}^j = \Phi^{-1} \left( F_{\varepsilon^j}(\varepsilon_{it}^j) \right)$ .

- ✓ If the Gaussian copula is assumed, the joint pdf of  $\varepsilon_{it}^1, \dots, \varepsilon_{it}^J$  is
$$f_\varepsilon(\varepsilon_{it}^1, \dots, \varepsilon_{it}^J)$$

$$= f_\varepsilon(\varepsilon_{it})$$

$$= c\left(\Phi^{-1}\left(F_{\varepsilon^1}(\varepsilon_{it}^1)\right), \dots, \Phi^{-1}\left(F_{\varepsilon^J}(\varepsilon_{it}^J)\right); R\right) \times \prod_{j=1}^J f_{\varepsilon^j}(\varepsilon_{it}^j).$$

- ✓ The log-likelihood function is

$$\ln L(\theta) = \sum_{i=1}^N \sum_{t=1}^T \ln f_\varepsilon(\varepsilon_{it}^1, \dots, \varepsilon_{it}^J).$$

The maximum likelihood (ML) estimator of  $\theta$  is defined as

$$\hat{\theta}_{\text{ML}} = \underset{\theta \in \Theta}{\text{argMax}} \ln L(\theta).$$

## The model with random effects

- ✓ Let  $\alpha_i = (\alpha_i^1, \dots, \alpha_i^J)'$  be the  $J \times 1$  vector of random effects
- ✓ Recall  $e_{it}^j = \alpha_i^j + v_{it}^j - u_{it}^j$  and  $\varepsilon_{it} = (\varepsilon_{it}^1, \dots, \varepsilon_{it}^J)'$ .

Let  $e_{it} = (e_{it}^1, \dots, e_{it}^J)'$  be a  $J \times 1$  vector and

$e_{i\cdot} = (e'_{i1}, \dots, e'_{iT})'$  be a  $JT \times 1$  vector.

- ✓  $e_{i1}^j, \dots, e_{iT}^j$  are correlated to each other due to the common component  $\alpha_i^j$ , but they are conditionally independent if  $\alpha_i^j$  is known.

- ✓ By definition,

$$f_e(e_{it}) = \int f_{e|\alpha}(e_{it}^1, \dots, e_{it}^J | \alpha_i) f_\alpha(\alpha_i) d\alpha_i,$$

where  $f_{e|\alpha}(e_{it}^1, \dots, e_{it}^J | \alpha_i) = f_\varepsilon(\varepsilon_{it}^1, \dots, \varepsilon_{it}^J)$ .

- ✓ It suggests that  $f_e(e_{it})$  can be evaluated by the **simulated joint pdf**

$$f_e^s(e_{it}) = \frac{1}{M} \sum_{m=1}^M f_\varepsilon(\varepsilon_{it(m)}),$$

where  $\varepsilon_{it(m)} = (\varepsilon_{it(m)}^1, \dots, \varepsilon_{it(m)}^J)'$  is a  $J \times 1$  vector,

$$\varepsilon_{it(m)}^j = e_{it}^j - \alpha_{i(m)}^j \text{ and}$$

$\alpha_{i(m)}^j$  denotes the  $m^{th}$  draw from the distribution of  $\alpha_i^j$ .

The superscript  $s$  of  $f_e^s(e_{it})$  denotes the simulated density.

- The simulated joint pdf  $f_e^s(e_{it})$  can be evaluated by

$$f_e^s(e_{it}) = \frac{1}{M} \sum_{m=1}^M \left[ c(F_{\varepsilon^1}(\varepsilon_{it(m)}^1), \dots, F_{\varepsilon^J}(\varepsilon_{it(m)}^J); R) \right] \\ \times \prod_{j=1}^J f_{\varepsilon^j}^s(\varepsilon_{it(m)}^j).$$

- ✓ For the special case when  $J = 1$ ,

$$f_{e^j}(e_{it}^j) = \int f_{e^j|\alpha^j}(e_{it}^j | \alpha_i^j) f_{\alpha^j}(\alpha_i^j) d\alpha_i^j,$$

which can be evaluated by the simulated density

$$f_{e^j}^s(e_{it}^j) = \frac{1}{M} \sum_{m=1}^M f_{\varepsilon^j}(e_{it}^j - \alpha_{i(m)}^j) = \frac{1}{M} \sum_{m=1}^M f_{\varepsilon^j}(\varepsilon_{it(m)}^j),$$

where  $\alpha_{i(m)}^j$  denotes the  $m^{\text{th}}$  Halton draw from  $\alpha_i^j$ 's distribution  $N(0, \sigma_{\alpha j}^2)$ .

- ✓ Consider the  $JT \times 1$  random vector  $e_{i\cdot} = (e'_{i1}, \dots, e'_{iT})'$ .

The joint pdf of  $e_{i\cdot}$  conditional on  $\alpha_i$  is

$$\begin{aligned}
 & f_{e|\alpha}(e_{i\cdot} | \alpha_i) \\
 &= f_{e|\alpha}(e_{i1}, \dots, e_{iT} | \alpha_i) \\
 &= f_\varepsilon(\varepsilon_{i\cdot}) \\
 &\quad (\text{Note: } e_{i1}, \dots, e_{iT} \text{ are conditionally independent given } \alpha_i.) \\
 &= \prod_{t=1}^T f_\varepsilon(\varepsilon_{it}) \\
 &= \prod_{t=1}^T f_\varepsilon(\varepsilon_{it}^1, \dots, \varepsilon_{it}^J).
 \end{aligned}$$

- ✓ The joint pdf of  $e_{i\cdot}$  can be approximated by the simulated density

$$\begin{aligned}
 f_e^s(e_{i\cdot}) &= \int f_{e|\alpha}^s(e_{i\cdot}|\alpha_i) f_\alpha(\alpha_i) d\alpha_i \\
 &= \frac{1}{M} \sum_{m=1}^M \left[ \prod_{t=1}^T f_\varepsilon(\varepsilon_{it(m)}^1, \dots, \varepsilon_{it(m)}^J) \right] \\
 &= \frac{1}{M} \sum_{m=1}^M \prod_{t=1}^T \left[ c(F_{\varepsilon^1}(\varepsilon_{it(m)}^1), \dots, F_{\varepsilon^J}(\varepsilon_{it(m)}^J); R) \right] \\
 &\quad \times \prod_{j=1}^J f_{\varepsilon^j}^s(\varepsilon_{it(m)}^j)
 \end{aligned}$$

- ✓ The logarithm of the simulated likelihood function of the SF system is

$$\ln L^s(\theta_1, \dots, \theta_J, R)$$

$$= \sum_{i=1}^N \ln f_e^s(e_{i..})$$

$$= \sum_{i=1}^N \ln \left\{ \frac{1}{M} \sum_{m=1}^M \prod_{t=1}^T \left[ c \left( F_{\varepsilon^1}(\varepsilon_{it(m)}^1), \dots, F_{\varepsilon^J}(\varepsilon_{it(m)}^J); R \right) \right] \times \prod_{j=1}^J f_{\varepsilon^j}^s(\varepsilon_{it(m)}^j) \right\}.$$

- ✓ Let  $\theta = (\theta'_1, \dots, \theta'_J, R)'$  be the set of all parameters and let  $\Theta$  denote the parameter space, then the maximum simulated likelihood (MSL) estimator of  $\theta$  is defined as

$$\hat{\theta}_{\text{MSL}} = \underset{\theta \in \Theta}{\text{argMax}} \ln L^s(\theta).$$

## Prediction of the inefficiency and technical efficiency

- ✓ For a given division  $j$ , let  $g(\cdot): \mathbb{R}^+ \rightarrow \mathbb{R}$  be a continuous function of  $u_{it}^j$  and we are interested in the conditional expectation  $\mathbb{E}(g(u_{it}^j)|e_{it}^j)$ .

- ✓ Examples:

If  $g(u_{it}^j) = u_{it}^j$ , then  $\mathbb{E}(g(u_{it}^j)|e_{it}^j) = \mathbb{E}(u_{it}^j|e_{it}^j)$ ;

if  $g(u_{it}^j) = e^{-u_{it}^j}$ , then  $\mathbb{E}(g(u_{it}^j)|e_{it}^j) = \mathbb{E}(e^{-u_{it}^j}|\varepsilon_{it}^j)$ .

- ✓ The conditional expectation of the inefficiency given the composite error  $e_{it}^j$  is defined as

$$\mathbb{E}(u_{it}^j | e_{it}^j) = \int_0^\infty u_{it}^j f_{u^j | e^j}(u_{it}^j | e_{it}^j) du_{it}^j,$$

where

$$\begin{aligned} & f_{u^j | e^j}(u_{it}^j | e_{it}^j) \\ &= \frac{\int_{-\infty}^\infty f_{u^j, e^j, \alpha^j}(u_{it}^j, e_{it}^j, \alpha_i^j) d\alpha_i^j}{f_{e^j}(e_{it}^j)} \\ &= \frac{\int_{-\infty}^\infty f_{u^j | e^j, \alpha^j}(u_{it}^j | e_{it}^j, \alpha_i^j) f_{e^j | \alpha^j}(e_{it}^j | \alpha_i^j) f_{\alpha^j}(\alpha_i^j) d\alpha_i^j}{\int_{-\infty}^\infty f_{e^j | \alpha^j}(e_{it}^j | \alpha_i^j) f_{\alpha^j}(\alpha_i^j) d\alpha_i^j}. \end{aligned}$$

- ✓ It follows that

$$\begin{aligned}
 & \mathbb{E}(u_{it}^j | e_{it}^j) \\
 &= \int_0^\infty u_{it}^j \int_{-\infty}^\infty f_{u^j | e^j, \alpha^j}(u_{it}^j | e_{it}^j, \alpha_i^j) \frac{f_{e^j | \alpha^j}(e_{it}^j | \alpha_i^j) f_{\alpha^j}(\alpha_i^j)}{\int_{-\infty}^\infty f_{e^j | \alpha^j}(e_{it}^j | \alpha_i^j) f_{\alpha^j}(\alpha_i^j) d\alpha_i^j} d\alpha_i^j du_{it}^j \\
 &= \int_{-\infty}^\infty \left( \int_0^\infty u_{it}^j f_{u^j | e^j, \alpha^j}(u_{it}^j | e_{it}^j, \alpha_i^j) du_{it}^j \right) \frac{f_{e^j | \alpha^j}(e_{it}^j | \alpha_i^j) f_{\alpha^j}(\alpha_i^j)}{\int_{-\infty}^\infty f_{e^j | \alpha^j}(e_{it}^j | \alpha_i^j) f_{\alpha^j}(\alpha_i^j) d\alpha_i^j} d\alpha_i^j
 \end{aligned}$$

- ✓ The above result is related to the law of iterative expectation

$$\mathbb{E}(u_{it}^j | e_{it}^j) = \mathbb{E}[\mathbb{E}(u_{it}^j | e_{it}^j, \alpha_i^j) | e_{it}^j].$$

- ✓ Given  $f(u_{it}^j | e_{it}^j, \alpha_i^j) = f(u_{it}^j | \varepsilon_{it}^j)$  and the well-known result

$$\mathbb{E}(u_{it}^j | \varepsilon_{it}^j) = \tilde{\mu}_{it}^j + \tilde{\sigma}_{it}^j \left[ \frac{\phi(-\tilde{\mu}_{it}^j / \tilde{\sigma}_{it}^j)}{1 - \Phi(-\tilde{\mu}_{it}^j / \tilde{\sigma}_{it}^j)} \right],$$

where  $\varepsilon_{it}^j = v_{it}^j - u_{it}^j$ ,  $\tilde{\mu}_{it}^j = -\varepsilon_{it}^j \sigma_{uj,it}^2 / (\sigma_{uj,it}^2 + \sigma_{vj}^2)$  and  $\tilde{\sigma}_{it}^{j2} = \sigma_{uj,it}^2 \sigma_{vj}^2 / (\sigma_{uj,it}^2 + \sigma_{vj}^2)$ , by assumptions [A2]-[A5].

- ✓ The simulated estimator  $\mathbb{E}^s(u_{it}^j | e_{it}^j)$  of  $\mathbb{E}(u_{it}^j | e_{it}^j)$ , which is defined as

$$\mathbb{E}^s(u_{it}^j | e_{it}^j) = \sum_{m=1}^M \left\{ \tilde{\mu}_{it(m)}^j + \tilde{\sigma}_{it}^j \left[ \frac{\phi(-\tilde{\mu}_{it(m)}^j / \tilde{\sigma}_{it}^j)}{1 - \Phi(-\tilde{\mu}_{it(m)}^j / \tilde{\sigma}_{it}^j)} \right] \right\} W_{it(m)}^j.$$

where  $\tilde{\mu}_{it(m)}^j = -\varepsilon_{it(m)}^j \sigma_{uj,it}^2 / (\sigma_{uj,it}^2 + \sigma_{vj}^2)$ ,  $\varepsilon_{it(m)}^j = e_{it}^j - \alpha_{i(m)}^j$

and the weight  $W_{it(m)}^j$  is defined as

$$W_{it(m)}^j = \frac{f_{\varepsilon^j}(e_{it}^j - \alpha_{i(m)}^j)}{\sum_{m=1}^M f_{\varepsilon^j}(e_{it}^j - \alpha_{i(m)}^j)} = \frac{f_{\varepsilon^j}(\varepsilon_{it(m)}^j)}{\sum_{m=1}^M f_{\varepsilon^j}(\varepsilon_{it(m)}^j)}.$$

- ✓ Note that the weights  $W_{it(m)}^j$  for  $m = 1, \dots, M$  have a sum equal to one.

✓ Moreover, given the result

$$\mathbb{E}\left(e^{-u_{it}^j}|\varepsilon_{it}^j\right) = \frac{1 - \Phi(\tilde{\sigma}_{it} - \tilde{\mu}_{it}^j/\tilde{\sigma}_{it}^j)}{1 - \Phi(-\tilde{\mu}_{it}^j/\tilde{\sigma}_{it}^j)} \exp\left(-\tilde{\mu}_{it}^j + \frac{1}{2}\tilde{\sigma}_{it}^{j2}\right),$$

and

$$\mathbb{E}\left(e^{-u_{it}^j}|e_{it}^j\right) = \mathbb{E}\left[\mathbb{E}\left(e^{-u_{it}^j}|e_{it}^j, \alpha_i^j\right)|e_{it}^j\right],$$

one can obtain the simulated estimator of TE,

$$\begin{aligned} & \mathbb{E}^s\left(e^{-u_{it}^j}|e_{it}^j\right) \\ &= \sum_{m=1}^M \left[ \frac{1 - \Phi(\tilde{\sigma}_{it}^j - \tilde{\mu}_{it(m)}^j/\tilde{\sigma}_{it}^j)}{1 - \Phi(-\tilde{\mu}_{it(m)}^j/\tilde{\sigma}_{it}^j)} \exp\left(-\tilde{\mu}_{it(m)}^j + \frac{1}{2}\tilde{\sigma}_{it}^{j2}\right) \right] W_{it(m)}^j. \end{aligned}$$

## Simulation

- ✓ The DGP:

$$y_{it}^1 = \beta_0^1 + x_{it}^1 \beta_1^1 + \alpha_i^1 + \varepsilon_{it}^1,$$

$$y_{it}^2 = \beta_0^2 + x_{it}^2 \beta_1^2 + \alpha_i^2 + \varepsilon_{it}^2,$$

where  $\varepsilon_{it}^j = v_{it}^j - u_{it}^j$ ,  $v_{it}^j \sim i.i.d. N(0, \sigma_{vj}^2)$ , and  $u_{it}^j \sim N^+(0, \sigma_{uj,it}^2)$  for  $j = 1, 2$ .

- ✓  $\alpha_i^1$  and  $\alpha_i^2$  are generated from a bivariate normal distribution with zero mean, correlation coefficient  $\varphi$ , and variance  $\sigma_{\alpha j}^2$ ,

where  $j = 1, 2$ .

- ✓ The heteroscedastic variance of  $u_{it}^j$  is specified as

$$\sigma_{uj,it} = \exp(\delta_0^j + \delta_1^j w_{it}^j).$$

- ✓ The true parameters are set as

$$\beta_0^1 = 0.75, \beta_1^1 = 0.75, \sigma_{\alpha 1} = 0.1, \sigma_{\nu 1} = 0.15, \delta_0^1 = -0.5, \delta_1^1 = 0.1,$$

$$\beta_0^2 = 1.5, \beta_1^2 = 0.5, \sigma_{\alpha 2} = 0.15, \sigma_{\nu 2} = 0.1, \delta_0^2 = -0.75, \delta_1^2 = 0.5.$$

## ■ **Experiment I:**

We investigate the sampling patterns of our estimator by considering the following different combinations of  $N$ ,  $T$  and  $\rho$ :

$$N = \{100, 200\}, T = \{5, 10\} \text{ and } \rho = \{0.25, 0.75\}.$$

## ■ Experiment II:

We reestimate the above model using **Clayton**, **Gumbel**, **FGM**  
**(Farlie-Gumbel-Morgenstern)** and **AMH (Ali-Mikhail-Haq)**  
copulas for the sample size  $N = 100$  and  $T = 5$  in order to  
investigate how the ML estimator performs under a misspecified  
copula.

- ✓ Both the Clayton and Gumbel copulas are Archimedean copulas

$$C_{Arch}(\zeta_1, \zeta_2) = \psi^{-1}[\psi(\zeta_1) + \psi(\zeta_2)],$$

where  $\psi: [0,1] \rightarrow \mathbb{R}^+$  is called the Archimedean generator.  $\psi(\cdot)$  is a continuous, strictly decreasing, and convex function and satisfies  $\psi(0) = \infty$  and  $\psi(1) = 0$ .

The two arguments  $\zeta_{j,it}$  are defined as  $\zeta_{j,it} = F_{\varepsilon^j}(\varepsilon_{it}^j)$  for  $j = 1, 2$ .  
The corresponding density of the Archimedean copula is

$$c_{Arch}(\zeta^1, \zeta^2) = \frac{-\psi''(C_{Arch}(\zeta^1, \zeta^2))\psi'(\zeta^1) + \psi'(\zeta^2)}{[\psi'(C_{Arch}(\zeta^1, \zeta^2))]^3}.$$

**(i).** If  $\psi(\zeta) = \zeta^{-a} - 1$ , we obtain the **Clayton copula**, which has the form

$$C_{Clay}(\zeta_1, \zeta_2) = (\zeta_1^{-a} + \zeta_2^{-a} - 1)^{-1/a},$$

where  $0 < a < \infty$  controls the strength of dependence. When  $a = 0$ , there is no dependence; and when  $a = \infty$ , there is perfect dependence.

**(ii).** If  $\psi(\zeta) = (-\ln \zeta)^a$ , we obtain the **Gumbel copula**

$$C_{Gum}(\zeta_1, \zeta_2) = \exp\{ -[(-\ln \zeta_1)^a + (-\ln \zeta_2)^a]^{1/a} \},$$

where  $a \geq 1$  controls the strength of dependence. When  $a = 1$ , there is no dependence; and when  $a = \infty$ , there is perfect dependence.

**(iii).** The third copula we considered is the **FGM** copula

$$C_{FGM}(\zeta_1, \zeta_2) = \zeta_1 \zeta_2 [1 + \kappa_F (1 - \zeta_1)(1 - \zeta_2)],$$

where  $-1 \leq \kappa_F \leq 1$  is the copula parameter and  $\zeta_1$  and  $\zeta_2$  are defined as before. The corresponding copula density is

$$c_{FGM}(\zeta_1, \zeta_2) = 1 + \kappa_F (1 - 2\zeta_1)(1 - 2\zeta_2).$$

The Spearman's  $\rho$  of the FGM copula is  $\kappa_F/3$ , so it ranges between  $-1/3$  and  $1/3$ .

**(iv).** The last copula is the **AMH** copula, whose copula function is defined as

$$C_{AMH}(\zeta_1, \zeta_2) = \frac{\zeta_1 \zeta_2}{1 - \kappa_A(1 - \zeta_1)(1 - \zeta_2)},$$

where  $-1 \leq \kappa_A \leq 1$  is the copula parameter and  $\zeta_1$  and  $\zeta_2$  are defined as before. The AMH copula density is

$$\begin{aligned} c_{AMH}(\zeta_1, \zeta_2) \\ = [1 + \kappa_A(\zeta_1 \zeta_2 + \zeta_1 + \zeta_2 - 2 + \kappa_A(1 - \zeta_1)(1 - \zeta_2))] \\ \times [1 - \kappa_A(1 - \zeta_1)(1 - \zeta_2)]^{-3}. \end{aligned}$$

- ✓ In the DGP, the true copula is a Gaussian copula and we consider  $\rho = 0.25$  and  $0.75$ .
- ✓ Since the dependence parameters of the Clayton, Gumbel, FGM and AMH copulas are not directly comparable with the Gaussian copula parameter  $\rho$ , we do not report the biases and RMSEs of the dependence parameters.

## ■ **Experiment III:**

- ✓ We allow correlation between the random effects, i.e.,  $\alpha_i^1$  and  $\alpha_i^2$  are correlated with correlation coefficient  $\varphi$ .

- ✓ **Main objective:**

To investigate how the degree of the correlation between the random effects affect the performance of the MSL estimator, w

- ✓ We consider **independent** and **Gaussian copulas** for

$$(N, T) = \{(100, 5), (200, 10)\}$$

$$\rho = \{0.25, 0.75\}$$

$$\varphi = \{0.25, 0.75\}$$

**Table 1: Biases of the Monte Carlo Experiments (DGP is Gaussian copula and  $\alpha_i$ 's are independent)**

N	T	Equation 1					Equation 2						
		$\beta_0^1$	$\beta_1^1$	$\sigma_{\alpha 1}$	$\sigma_{v1}$	$\delta_0^1$	$\delta_1^1$	$\beta_0^2$	$\beta_1^2$	$\sigma_{\alpha 2}$	$\sigma_{v2}$	$\delta_0^2$	$\delta_1^2$
<b>A. Independent copula</b>													
								$\rho = 0.25$					
100	5	0.0342	-0.0030	0.0198	-0.0011	-0.0205	-0.0101	-0.0011	-0.0032	0.0113	-0.0027	-0.0040	0.0075
100	10	0.0136	-0.0001	0.0187	0.0009	-0.0021	0.0034	-0.0115	-0.0001	0.0130	-0.0010	-0.0034	0.0116
200	5	0.0034	-0.0001	0.0130	0.0015	0.0107	0.0185	-0.0090	-0.0008	-0.0093	-0.0004	-0.0041	0.0027
200	10	0.0128	-0.0012	0.0134	-0.0005	-0.0028	-0.0026	-0.0137	0.0005	-0.0077	0.0007	-0.0033	0.0024
								$\rho = 0.75$					
100	5	0.0248	-0.0019	0.0189	0.0008	-0.0312	-0.0173	-0.0026	-0.0029	0.0113	-0.0011	-0.0083	0.0096
100	10	0.0146	0.0001	0.0178	-0.0002	-0.0031	0.0002	-0.0157	0.0012	0.0138	-0.0007	-0.0005	0.0056
200	5	0.0054	-0.0003	0.0133	0.0000	0.0106	0.0145	-0.0081	-0.0008	-0.0097	-0.0010	-0.0024	0.0087
200	10	0.0131	-0.0013	0.0135	0.0001	-0.0028	-0.0006	-0.0115	0.0002	-0.0077	-0.0003	-0.0009	-0.0035
<b>B. Gaussian copula</b>													
								$\rho = 0.25$					
100	5	-0.0250	0.0023	0.0017	-0.0002	-0.0279	-0.0175	0.0301	-0.0002	0.0080	-0.0012	-0.0111	0.0184
100	10	-0.0066	0.0001	0.0023	0.0003	-0.0063	-0.0003	0.0295	0.0009	0.0070	-0.0017	-0.0016	0.0051
200	5	-0.0016	0.0011	0.0009	0.0012	0.0043	0.0081	0.0166	-0.0016	0.0006	-0.0002	-0.0022	0.0069
200	10	-0.0033	0.0014	0.0038	0.0007	0.0021	0.0064	0.0141	-0.0008	0.0016	0.0004	-0.0004	0.0051
								$\rho = 0.75$					
100	5	-0.0126	0.0010	0.0049	-0.0024	-0.0070	-0.0031	0.0347	-0.0010	0.0085	-0.0034	-0.0047	0.0030
100	10	-0.0087	0.0006	0.0051	-0.0015	-0.0036	-0.0005	0.0298	0.0003	0.0094	-0.0005	-0.0052	0.0095
200	5	0.0032	0.0007	0.0007	0.0014	-0.0021	0.0007	0.0145	-0.0011	-0.0009	0.0014	-0.0037	0.0065
200	10	0.0007	0.0010	0.0021	0.0008	-0.0027	0.0014	0.0120	-0.0005	-0.0007	0.0006	-0.0028	0.0010

Notes: The total number of replications is 500. N/A denotes not applicable.

**Table 2: RMSEs of the Monte Carlo Experiments (DGP is Gaussian copula and  $\alpha_i$ 's are independent)**

N	T	Equation 1						Equation 2						
		$\beta_0^1$	$\beta_1^1$	$\sigma_{\alpha 1}$	$\sigma_{v1}$	$\delta_0^1$	$\delta_1^1$	$\beta_0^2$	$\beta_1^2$	$\sigma_{\alpha 2}$	$\sigma_{v2}$	$\delta_0^2$	$\delta_1^2$	
<b>A. Independent copula</b>														
$\rho = 0.25$														
100	5	0.0750	0.0095	0.0200	0.0271	0.0920	0.0694	0.0510	0.0137	0.0135	0.0286	0.0541	0.0925	
100	10	0.0548	0.0070	0.0117	0.0154	0.0602	0.0484	0.0369	0.0097	0.0087	0.0130	0.0379	0.0595	
200	5	0.0571	0.0071	0.0146	0.0179	0.0620	0.0484	0.0361	0.0096	0.0102	0.0161	0.0386	0.0675	
200	10	0.0396	0.0050	0.0081	0.0113	0.0443	0.0351	0.0240	0.0062	0.0060	0.0098	0.0264	0.0447	
$\rho = 0.75$														
100	5	0.0750	0.0095	0.0196	0.0249	0.0908	0.0656	0.0510	0.0137	0.0142	0.0245	0.0588	0.0903	
100	10	0.0548	0.0068	0.0121	0.0164	0.0588	0.0475	0.0343	0.0090	0.0087	0.0129	0.0371	0.0627	
200	5	0.0581	0.0071	0.0139	0.0179	0.0604	0.0516	0.0370	0.0097	0.0102	0.0166	0.0397	0.0579	
200	10	0.0389	0.0051	0.0081	0.0111	0.0433	0.0333	0.0253	0.0065	0.0062	0.0092	0.0265	0.0443	
<b>B. Gaussian copula</b>														
$\rho = 0.25$														
100	5	0.0793	0.0100	0.0231	0.0257	0.0841	0.0617	0.0526	0.0136	0.0130	0.0245	0.0572	0.0902	
100	10	0.0528	0.0068	0.0120	0.0147	0.0619	0.0482	0.0345	0.0090	0.0090	0.0128	0.0364	0.0616	
200	5	0.0501	0.0062	0.0152	0.0174	0.0630	0.0499	0.0346	0.0093	0.0088	0.0164	0.0387	0.0587	
200	10	0.0377	0.0047	0.0088	0.0112	0.0420	0.0343	0.0237	0.0063	0.0063	0.0094	0.0253	0.0406	
$\rho = 0.75$														
100	5	0.0582	0.0070	0.0182	0.0233	0.0736	0.0473	0.0400	0.0098	0.0103	0.0201	0.0532	0.0650	
100	10	0.0386	0.0047	0.0112	0.0141	0.0496	0.0330	0.0267	0.0064	0.0094	0.0117	0.0342	0.0454	
200	5	0.0394	0.0047	0.0118	0.0148	0.0520	0.0377	0.0270	0.0067	0.0069	0.0138	0.0349	0.0426	
200	10	0.0295	0.0036	0.0069	0.0096	0.0331	0.0243	0.0180	0.0044	0.0052	0.0081	0.0236	0.0292	

Notes: The total number of replications is 500. N/A denotes not applicable.

**Table 3: Bias and RMSE for different copulas ( $N = 100$ ,  $T = 5$ , DGP is Gaussian copula and  $\alpha_i$ 's are independent)**

Copula	$\rho$	Equation 1						Equation 2					
		$\beta_0^1$	$\beta_1^1$	$\sigma_{\alpha 1}$	$\sigma_{v1}$	$\delta_0^1$	$\delta_1^1$	$\beta_0^2$	$\beta_1^2$	$\sigma_{\alpha 2}$	$\sigma_{v2}$	$\delta_0^2$	$\delta_1^2$
<b>Bias</b>												<b>Bias</b>	
Gaussian	0.25	-0.0250	0.0023	0.0017	-0.0002	-0.0279	-0.0175	0.0301	-0.0002	0.0080	-0.0012	-0.0111	0.0184
Indep.	0.25	0.0342	-0.0030	0.0198	-0.0011	-0.0205	-0.0101	-0.0011	-0.0032	0.0113	-0.0027	-0.0040	0.0075
Clayton	0.25	-0.0205	0.0022	0.0044	-0.0048	-0.0149	-0.0119	0.0299	0.0003	0.0078	-0.0029	-0.0075	0.0161
Gumbel	0.25	-0.2478	0.0022	0.0053	0.0028	-0.0308	-0.0112	0.0270	0.0002	0.0086	0.0022	-0.0195	0.0175
FGM	0.25	-0.0215	0.0020	0.0031	-0.0020	-0.0189	-0.0096	0.0301	-0.0005	0.0074	0.0001	-0.0128	0.0204
AMH	0.25	0.0390	0.0021	-0.0003	-0.0127	0.0058	-0.0182	0.0734	0.0005	0.0081	-0.0116	0.0113	0.0033
Gaussian	0.75	-0.0126	0.0010	0.0049	-0.0024	-0.0070	-0.0031	0.0347	-0.0010	0.0085	-0.0034	-0.0047	0.0030
Indep.	0.75	0.0248	-0.0019	0.0189	0.0008	-0.0312	-0.0173	-0.0026	-0.0029	0.0113	-0.0011	-0.0083	0.0096
Clayton	0.75	0.0126	0.0017	0.0222	-0.0367	0.0556	-0.0088	0.0524	-0.0001	0.0130	-0.0273	0.0502	0.0024
Gumbel	0.75	-0.0372	0.0012	0.0054	0.0330	-0.0661	0.0039	0.0195	-0.0007	0.0113	0.0230	-0.0509	0.0209
FGM	0.75	-0.0375	0.0016	0.0071	-0.0022	-0.0596	-0.0054	0.0153	-0.0001	0.0085	-0.0013	-0.0546	0.0203
AMH	0.75	0.0613	0.0016	-0.0027	-0.0039	0.0285	-0.0138	0.0808	-0.0008	0.0082	-0.0091	0.0342	-0.0074
<b>RMSE</b>												<b>RMSE</b>	
Gaussian	0.25	0.0793	0.0100	0.0231	0.0257	0.0841	0.0617	0.0526	0.0136	0.0130	0.0245	0.0572	0.0902
Indep.	0.25	0.0750	0.0095	0.0200	0.0271	0.0920	0.0694	0.0510	0.0137	0.0135	0.0286	0.0541	0.0925
Clayton	0.25	0.0772	0.0094	0.0212	0.0265	0.0891	0.0658	0.0517	0.0133	0.0135	0.0245	0.0599	0.0897
Gumbel	0.25	0.0095	0.0095	0.0214	0.0273	0.0907	0.0655	0.0525	0.0133	0.0129	0.0272	0.0593	0.0906
FGM	0.25	0.0755	0.0094	0.0221	0.0260	0.0916	0.0681	0.0519	0.0132	0.0130	0.0251	0.0580	0.0862
AMH	0.25	0.0819	0.0104	0.0252	0.0304	0.0826	0.0625	0.0593	0.0156	0.0146	0.0279	0.0512	0.0848
Gaussian	0.75	0.0582	0.0070	0.0182	0.0233	0.0736	0.0473	0.0400	0.0098	0.0103	0.0201	0.0532	0.0214
Indep.	0.75	0.0750	0.0095	0.0196	0.0249	0.0908	0.0656	0.0510	0.0137	0.0142	0.0245	0.0588	0.0903
Clayton	0.75	0.0639	0.0078	0.0163	0.0235	0.0709	0.0474	0.0420	0.0108	0.0124	0.0199	0.0446	0.0687
Gumbel	0.75	0.0606	0.0072	0.0172	0.0249	0.0821	0.0534	0.0442	0.0103	0.0109	0.0278	0.0724	0.0706
FGM	0.75	0.0647	0.0079	0.0189	0.0239	0.0837	0.0636	0.0486	0.0122	0.0124	0.0215	0.0571	0.0787
AMH	0.75	0.0925	0.0120	0.0267	0.0362	0.0846	0.0656	0.0650	0.0169	0.0164	0.0337	0.0555	0.0858

Note: The total number of replications is 500.

**Table 4: Bias when  $\alpha_i$ 's are correlated (DGP is Gaussian copula)**

				Equation 1				Equation 2								
N	T	$\varphi$		$\beta_0^1$	$\beta_1^1$	$\sigma_{\alpha 1}$	$\sigma_{v1}$	$\delta_0^1$	$\delta_1^1$	$\beta_0^2$	$\beta_1^2$	$\sigma_{\alpha 2}$	$\sigma_{v2}$	$\delta_0^2$	$\delta_1^2$	$\rho$
<b>Independent copula</b>																
																$\rho = 0.25$
100	5	0.25	-0.0268	0.0026	0.0017	-0.0007	-0.0285	-0.0184	0.0204	0.0015	0.0044	0.0007	-0.0121	0.0187	N/A	
200	10	0.25	-0.0044	0.0018	0.0034	0.0003	0.0027	0.0048	0.0179	-0.0005	0.0077	-0.0006	-0.0002	-0.0002	N/A	
100	5	0.75	-0.0237	0.0022	-0.0005	0.0008	-0.0229	-0.0117	0.0197	0.0016	0.0049	0.0010	-0.0143	0.0134	N/A	
200	10	0.75	-0.0047	0.0018	0.0047	0.0002	-0.0019	0.0016	0.0178	-0.0002	0.0117	0.0007	-0.0036	0.0009	N/A	
																$\rho = 0.75$
100	5	0.25	-0.0263	0.0025	-0.0017	0.0031	-0.0277	-0.0154	-0.0013	0.0037	-0.0005	0.0015	-0.0154	0.0141	N/A	
200	10	0.25	-0.0065	0.0020	0.0041	0.0007	-0.0005	0.0024	0.0190	-0.0005	0.0126	0.0008	-0.0034	0.0028	N/A	
100	5	0.75	-0.0270	0.0032	0.0018	-0.0037	-0.0135	-0.0105	0.0173	0.0026	0.0047	0.0002	-0.0142	0.0185	N/A	
200	10	0.75	-0.0077	0.0021	0.0035	0.0002	-0.0022	0.0016	0.0167	-0.0003	0.0078	-0.0005	-0.0011	0.0045	N/A	
<b>Gaussian copula</b>																
																$\rho = 0.25$
100	5	0.25	-0.0233	0.0023	-0.0043	0.0020	-0.0208	-0.0098	0.0264	0.0002	0.0011	0.0005	-0.0102	0.0157	0.0102	
200	10	0.25	-0.0059	0.0019	0.0009	0.0013	0.0008	0.0018	0.0154	-0.0003	0.0041	0.0013	-0.0030	0.0042	0.0050	
100	5	0.75	-0.0238	0.0023	-0.0177	0.0066	-0.0153	-0.0068	0.0006	0.0037	-0.0132	0.0078	-0.0117	0.0187	0.0391	
200	10	0.75	-0.0054	0.0019	-0.0037	0.0029	0.0002	0.0025	0.0171	0.0000	0.0039	0.0036	-0.0012	0.0052	0.0146	
																$\rho = 0.75$
100	5	0.25	-0.0126	0.0010	-0.0087	0.0032	-0.0103	-0.0031	0.0250	0.0006	-0.0029	0.0042	-0.1035	0.0136	0.0054	
200	10	0.25	0.0016	0.0009	-0.0058	0.0032	-0.0081	0.0012	0.0153	-0.0007	-0.0033	0.0040	-0.0037	0.0033	0.0023	
100	5	0.75	-0.0166	0.0019	-0.0638	0.0191	0.0041	-0.0014	0.0113	0.0014	-0.0391	0.0221	-0.0019	0.0116	0.0208	
200	10	0.75	0.0037	0.0008	0.0291	0.0108	0.0035	0.0017	0.0182	0.0021	-0.0200	0.0114	0.0011	0.0007	0.0098	

Notes: The total number of replications is 500. N/A denotes not applicable.  $\varphi$  is the correlation coefficient of  $\alpha_i^1$  and  $\alpha_i^2$

**Table 5: RMSE when  $\alpha_i$ 's are correlated (DGP is Gaussian copula)**

N	T	$\varphi$	Equation 1					Equation 2					$\rho$	
			$\beta_0^1$	$\beta_1^1$	$\sigma_{\alpha 1}$	$\sigma_{v1}$	$\delta_0^1$	$\delta_1^1$	$\beta_0^2$	$\beta_1^2$	$\sigma_{\alpha 2}$	$\sigma_{v2}$		
<b>Independent copula</b>														
									$\rho = 0.25$					
100	5	0.25	0.0820	0.0103	0.0253	0.0266	0.0903	0.0719	0.0563	0.0143	0.0144	0.0241	0.0579	0.0897
200	10	0.25	0.0396	0.0050	0.0091	0.0109	0.0449	0.0351	0.0249	0.0065	0.0066	0.0094	0.0261	0.0435
100	5	0.75	0.0821	0.0104	0.0236	0.0261	0.0864	0.0669	0.0543	0.0142	0.0132	0.0238	0.0605	0.0943
200	10	0.75	0.0396	0.0050	0.0082	0.0110	0.0447	0.0352	0.0261	0.0068	0.0067	0.0098	0.0266	0.0423
									$\rho = 0.75$					
100	5	0.25	0.0805	0.0103	0.0254	0.0250	0.0880	0.0678	0.0570	0.0147	0.0132	0.0234	0.0580	0.0888
200	10	0.25	0.0383	0.0049	0.0099	0.0114	0.0449	0.0361	0.0244	0.0063	0.0075	0.0093	0.0256	0.0421
100	5	0.75	0.0783	0.0099	0.0220	0.0258	0.0837	0.0670	0.0540	0.0137	0.0136	0.0238	0.0572	0.0942
200	10	0.75	0.0396	0.0045	0.0090	0.0114	0.0446	0.0350	0.0251	0.0068	0.0069	0.0095	0.0249	0.0436
<b>Gaussian copula</b>														
									$\rho = 0.25$					
100	5	0.25	0.0762	0.0097	0.0255	0.0250	0.0900	0.0653	0.0509	0.0132	0.0129	0.0240	0.0559	0.0861
200	10	0.25	0.0367	0.0047	0.0086	0.0111	0.0425	0.0347	0.0237	0.0062	0.0062	0.0089	0.0250	0.0405
100	5	0.75	0.0761	0.0094	0.0280	0.0251	0.0873	0.0660	0.0533	0.0135	0.0145	0.0237	0.0560	0.0886
200	10	0.75	0.0370	0.0047	0.0098	0.0120	0.0440	0.0326	0.0239	0.0062	0.0069	0.0094	0.0257	0.0419
									$\rho = 0.75$					
100	5	0.25	0.0588	0.0072	0.0178	0.0225	0.0723	0.0485	0.0398	0.0097	0.0095	0.0209	0.0540	0.0603
200	10	0.25	0.0269	0.0032	0.0077	0.0097	0.0354	0.0254	0.0187	0.0046	0.0053	0.0080	0.0241	0.0289
100	5	0.75	0.0553	0.0067	0.0281	0.0211	0.0676	0.0454	0.0408	0.0098	0.0100	0.0191	0.0540	0.0624
200	10	0.75	0.0283	0.0033	0.0107	0.0097	0.0346	0.0240	0.0185	0.0046	0.0062	0.0081	0.0250	0.0294

Notes: The total number of replications is 500. N/A denotes not applicable.  $\varphi$  is the correlation coefficient of  $\alpha_i^1$  and  $\alpha_i^2$ .

## The empirical application - Taiwan's international hotels

- ✓ Each hotel has two divisions, accommodation and restaurant, i.e.,  $J = 2$  for the system.
- ✓ The data is derived from the annual report of the Taiwan Tourism Bureau at the Ministry of Transportation and Communications. Our sample is an unbalanced panel data, which contains 725 sample observations from 61 international grand hotels during 2001-2013. The minimum observed time period is 6 years while the maximum period is 13 years.
- ✓ The accommodation division:  
**Output:** total revenue ( $y_1$ )  
**Inputs:** the total number of workers ( $x_{11}$ ), the total number of rooms ( $x_{12}$ ), and other expenses ( $x_{13}$ ), which include utilities, materials, maintenance fees, and so on.

- ✓ The restaurant division:
  - Output: the total revenue ( $y_2$ )
  - Inputs: the total number of workers ( $x_{21}$ ), the floor area of the restaurant ( $x_{22}$ ), and other expenses ( $x_{23}$ ), including utilities, materials, and so on.
- ✓ Logarithms are applied to outputs and inputs.
- ✓ The exogenous determinants of the inefficiencies of the two divisions:
  - Scale of the hotel ( $w_1$ ) - the scale variable  $w_1$  ranges from 1 to 5
  - Area dummy variable ( $w_2$ ) - one if the hotel is located in a scenic area and zero otherwise.
- ✓ Since these two divisions of a hotel share certain common characteristics, such as the same DMU, brand, location, etc.; we expect the two outputs  $y_1$  and  $y_2$  should be correlated to each other and the composite errors to be also.

- ✓ The empirical model is specified as

$$y_{it}^j = \beta_0^j + \beta_1^j x_{1,it}^j + \beta_2^j x_{2,it}^j + \beta_3^j x_{3,it}^j + \alpha_i^j + \varepsilon_{it}^j, \quad j = 1, 2.$$

- ✓ Under the Gaussian copula assumption, the joint pdf of  $\varepsilon_{it}^1$  and  $\varepsilon_{it}^2$  is

$$f_\varepsilon(\varepsilon_{it}^1, \varepsilon_{it}^2) = c(F_{\varepsilon^1}(\varepsilon_{it}^1), F_{\varepsilon^2}(\varepsilon_{it}^2)) f_{\varepsilon^1}(\varepsilon_{it}^1) f_{\varepsilon^2}(\varepsilon_{it}^2),$$

where

$$c(\zeta_{it}^1, \zeta_{it}^2) = \frac{1}{\sqrt{1-\rho^2}} \exp \left( \frac{(\zeta_{it}^1)^2 + (\zeta_{it}^2)^2}{2} + \frac{2\rho\zeta_{it}^1\zeta_{it}^2 - (\zeta_{it}^1)^2 - (\zeta_{it}^2)^2}{2(1-\rho^2)} \right),$$

copula, the linear correlation between  $F_{\varepsilon^1}$  and  $F_{\varepsilon^2}$  is

$$\gamma_{\text{Spearman}} = \frac{6}{\pi} \arcsin \frac{\rho}{2}.$$

**Table 6: Empirical results**

I. Separate Estimation		II. Joint Estimation Independent copula		III. Joint Estimation Gaussian Copula		I. Separate Estimation		II. Joint Estimation Independent copula		III. Joint Estimation Gaussian Copula	
coef.	s.e.	coef.	s.e.	coef.	s.e.	coef.	s.e.	coef.	s.e.	coef.	s.e.
<b>Accommodation Division:</b>						<b>Restaurant Division:</b>					
$\beta_1^1$	0.3104	0.0086 ***	0.3426	0.0214 ***	0.3000	0.0003 ***	$\beta_1^2$	0.4295	0.0083 ***	0.4611	0.0087 ***
$\beta_2^1$	0.4276	0.0117 ***	0.5229	0.0183 ***	0.3958	0.0011 ***	$\beta_2^2$	0.0442	0.0021 ***	0.0451	0.0018 ***
$\beta_3^1$	0.2835	0.0044 ***	0.2929	0.0040 ***	0.4411	0.0012 ***	$\beta_3^2$	0.1627	0.0049 ***	0.1566	0.0046 ***
$\beta_0^1$	13.0459	0.0583 ***	12.3334	0.0376 ***	11.9497	0.0029 ***	$\beta_0^2$	15.1821	0.0815 ***	15.104	0.0704 ***
<b>Random components :</b>						<b>Random components :</b>					
$\sigma_{\alpha 1}$	0.2808	0.0032 ***	0.3309	0.0034 ***	0.3244	0.0006 ***	$\sigma_{\alpha 2}$	0.4038	0.0075 ***	0.4855	0.0096 ***
$\sigma_{\nu 1}$	0.0766	0.0016 ***	0.0758	0.0018 ***	0.0021	0.0004 ***	$\sigma_{\nu 2}$	0.1373	0.0018 ***	0.1377	0.0018 ***
$\sigma_{u1} = \exp(\delta_1^1 w_1 + \delta_2^1 w_2 + \delta_0^1)$						$\sigma_{u2} = \exp(\delta_1^2 w_1 + \delta_2^2 w_2 + \delta_0^2)$					
$\delta_1^1$	-0.0863	0.0093 ***	-0.0872	0.0080 ***	-0.0759	0.0033 ***	$\delta_1^2$	-1.9918	0.0759 ***	-1.9702	0.0907 ***
$\delta_2^1$	-0.0304	0.0220 ***	-0.0583	0.0184 ***	-0.0549	0.0117 ***	$\delta_2^2$	-1.8949	0.0930 ***	-1.8748	0.0940 ***
$\delta_0^1$	-1.0901	0.0344 ***	-1.0606	0.0299 ***	-0.8321	0.0127 ***	$\delta_0^2$	2.1571	0.1309 ***	2.1193	0.1588 ***
<b>Copula parameter:</b>											
$\rho$	N/A		N/A		0.8431	0.0068 ***					
<b>Log-likelihood value:</b>											
lnL	362.0287		356.0293		581.9081						

**Table 7: Predicted Inefficiencies, TEs and marginal effects**

	Accommodation		Restaurant	
	mean	s.d.	mean	s.d.
<b>Panel A. Inefficiency <math>\mathbb{E}(u_{it}^j   \varepsilon_{it}^j)</math>:</b>				
I. Separate estimation	0.2071	0.0897	0.0874	0.2795
II. Independent copula	0.2130	0.0701	0.0848	0.2795
III. Gaussian copula	0.2506	0.0886	0.1301	0.2023
<b>Panel B. TE <math>\mathbb{E}(e^{-u_{it}^j}   \varepsilon_{it}^j)</math>:</b>				
I. Separate estimation	0.8231	0.0667	0.9356	0.1278
II. Independent copula	0.8181	0.0523	0.9404	0.1243
III. Gaussian copula	0.7914	0.0629	0.8949	0.1171
<b>Panel C. Marginal effects:</b>				
$10 \times \partial \mathbb{E}(u_{it}^j   \varepsilon_{it}^j) / \partial w_1$				
I. Separate estimation	-0.0681	0.0305	-0.6842	1.0502
II. Independent copula	-0.0660	0.0243	-0.6908	1.0213
III. Gaussian copula	-0.0002	0.0006	-0.6459	0.5155
$10 \times \Delta \mathbb{E}(u_{it}^j   \varepsilon_{it}^j) / \Delta w_2$				
I. Separate estimation	-0.0469	0.0307	-0.0520	0.0214
II. Independent copula	-0.0952	0.0487	-0.8343	2.0495
III. Gaussian copula	-0.1012	0.0404	-0.7774	0.7399

**Thank you !**